

Multidimensional Assessment of CCSSM

Measuring students' understanding of math concepts in this manner offers insight into the robustness of their knowledge, particularly of the Common Core State Standards for Mathematics.

By Sarah K. Bleiler and Denisse R. Thompson

Given the recent release and widespread state adoption of the Common Core State Standards for Mathematics (CCSSM), teachers across the country now have common goals related to what their students should understand about particular mathematical topics. The authors of CCSSM explain, "Asking a student to understand something means asking a teacher to assess whether the student has understood it" (CCSSI 2010, p. 4). As teachers, we know that although students may be able to perform a computation, state a definition, or provide an example, they may not actually understand the mathematics. So, what does it mean for students to understand mathematics, and how can teachers gauge this understanding?





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Researchers and educators have proposed several frameworks to use when thinking about what it means to “understand” mathematics. For example, the authors of *Adding It Up* (NRC 2001) suggested five interrelated strands of mathematical proficiency:

1. Conceptual understanding
2. Procedural fluency
3. Strategic competence
4. Adaptive reasoning
5. Productive disposition

Although such a framework is useful for thinking about the different competencies we would like our students to have as a result of their mathematical learning experiences, how this framework could be used as a tool for creating or modifying assessment tasks to meet each proficiency is not immediately obvious. We propose adapting a model originally used for curriculum development (Usiskin 2007) as a multidimensional approach to assessing students’ mathematical understanding across four dimensions:

1. Skills: Procedures, such as carrying out one-step or multiple-step algorithms, inventing new algorithms, and using technology to perform mathematical calculations
2. Properties: The underlying theories and principles of mathematics, often requiring students to identify or apply mathematical properties or provide mathematical justifications
3. Uses: Real-world applications, which often means expecting students to develop models to describe the mathematics
4. Representations: Visual depictions of mathematical concepts, such as graphs, pictures of geometric figures, or statistical plots

This multidimensional approach to understanding, described by the acronym SPUR (Usiskin 2007), can give teachers useful information about the depth of their students’ understanding of a mathematical topic. In fact, research has shown that if students have a rich network of connected ideas about a mathematical concept, their capacity to problem solve and succeed in novel mathematical situations is enhanced (NCTM 2000; NMAP 2008). Assessments encompassing all four of the SPUR



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dimensions give insight into the robustness of students’ mathematical understanding.

In the following section, we briefly examine the results from an international research project on elementary school students’ mathematical attainment in relation to SPUR (Bleiler and Thompson 2010; Thompson and Kaur 2011). Then we demonstrate how teachers might use SPUR as a guide for assessment in their classrooms, specifically in relation to CCSSM.

Using SPUR to analyze U.S. IPMA results

The International Project on Mathematical Attainment (IPMA) was a multinational, longitudinal study designed to assess the mathematical attainment of children throughout their elementary school years (K–grade 5). Although seventeen countries participated in the original study, the focus of our comments is the U.S. sample, demonstrating the importance of a multidimensional approach to assessment of students’ mathematical understanding. (For more information about the larger international IPMA study conducted by the Centre for Innovation in Mathematics Teaching, go to <http://www.cimt.plymouth.ac.uk/projects/ipma/default.htm>.) In the last year of the IPMA study, 181 fifth-grade students from six U.S. schools



A rich set of curriculum standards is not enough: Quality instruction must be paired with dynamic, multifaceted assessment that allows insight into students' understanding.

completed a 140-item mathematics test covering a range of content strands in the elementary school curriculum. Although the IPMA tests were not designed with SPUR in mind, researchers from several countries thought they might learn more about students' mathematical understanding by using SPUR to analyze the test results. Therefore, they categorized each test item as a Skills, Properties, Uses, or Representations item and then analyzed student achievement in each category (Bleiler and Thompson 2010; Thompson and Kaur 2011). The four tasks

in figure 1 are examples of items from each of the four SPUR categories on the IPMA fifth-grade test, listed in order according to their *S*, *P*, *U*, or *R* classification. These tasks could be used to assess students' understanding of decimals, one of three critical focus areas in grade 5 as outlined in CCSSM.

On the basis of the IPMA test results, what could we say about these students' *understanding* of this essential focus area? The majority of students (61 percent) could perform the computations necessary to solve the first problem, involving the multiplication of a two-digit whole number by a decimal, thus demonstrating an understanding of Skills needed to multiply decimals. However, only 42 percent of the students were able to accurately locate a decimal on a number line divided into hundredths, a task that would require an understanding related to Representations. Few students (33 percent) could successfully complete a two-step word problem involving money in decimal units (Uses), and even fewer (17 percent) were able to identify a decimal between two given decimals, thus demonstrating a weak understanding of the more abstract mathematical Properties related to decimals.

Using SPUR as an assessment guide

The examples from the IPMA test consider only one item from each dimension and therefore offer little information about students' overall

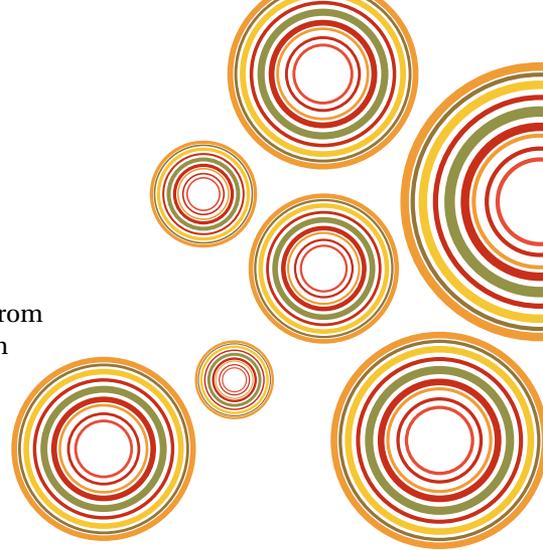
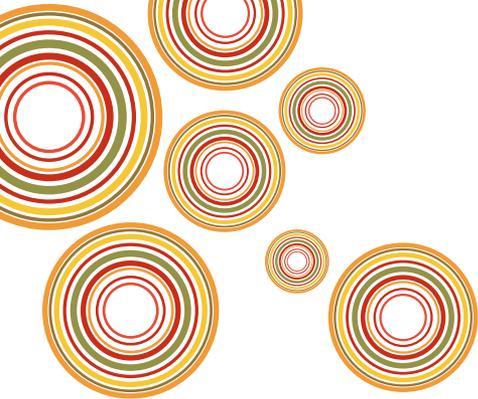


FIGURE 1

Consider using tasks like the following four items to assess students' understanding of decimal concepts. What would students' performance on these four tasks tell you about their depth of understanding of decimals?

1. $70 \times 0.3 = \underline{\quad}$.
2. Give a number to two decimal places, which lies between 122.257 and 122.263.
3. 6 tickets cost \$2.10. What is the cost of 13 tickets?
4. Show the position of 2.13 on the number line below:





understanding in each of the four dimensions. However, they do suggest that analyzing the results of assessments across the four SPUR dimensions can yield useful information about which dimensions of understanding need greater instructional focus for a particular mathematical topic. In this section, we illustrate how to use SPUR as a guide to create balanced assessments of students' mathematical understanding, particularly related to CCSSM.

A grade 2 example

In elementary school, K–grade 5, one of the major domains of focus is Number and Operations in Base Ten. At grade 2, a specific Standard (2.NBT) under this domain is for students to—

Understand place value.

1. Understand that the three digits of a three-digit number represent amounts of

hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

- a. 100 can be thought of as a bundle of ten tens—called a “hundred.”
- b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones). (CCSSI 2010, p. 19)

What might you expect students to demonstrate about this base-ten Standard during an assessment? From a quick scan of curricular materials and assessments related to place value, we believe the majority of tasks could be classified under the Properties dimension of SPUR because the tasks focus on the relationships of the digits relative to their placement in the numeral. Examples include tasks such as these:

- I am a number with 5 hundreds and 6 ones. What am I?
- What does the 4 in 341 represent?
- Write the digit in the hundreds place of 125.

Assessing multiple dimensions

To assess the Skills dimension of understanding in SPUR, you might ask students to mentally find the sum of 230 and 590. If a student has a strong understanding of place-value concepts, he or she should be able to mentally add the 2 hundreds and the 5 hundreds to get 7 hundreds as well as add the 3 tens and the 9 tens to get 12 tens. If the student understands the special case (a) of the Standard, then he or she should also recognize that 12 tens is the same as 1 hundred (a bundle of 10 tens) and 2 tens. Therefore, the student could mentally put together 7 hundreds and 1 hundred to get 8 hundreds and add the remaining 2 tens for a final answer of 820. However, a student with a fragile place-value understanding of three-digit numbers may have trouble solving this problem mentally and instead rely on customary written algorithms to complete the task.

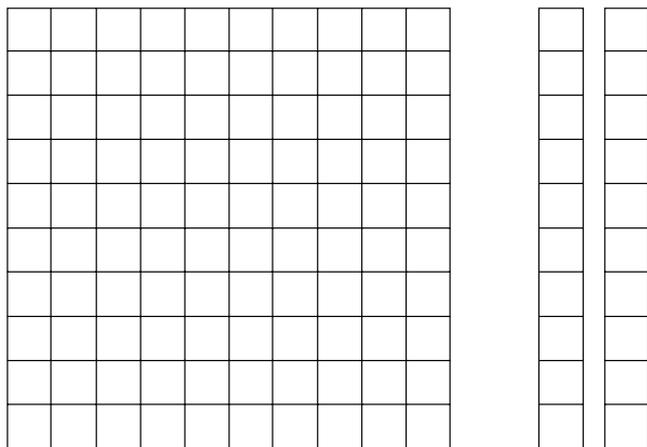
To assess the Uses dimension, think of a real-life situation that you could use to illustrate place-value concepts, such as the following task, which can be used to assess students' under-

FIGURE 2

These two Representation assessment tasks align with place-value concepts.

(a) Task 1

What would you need to add to the picture below to show the number 138?



(b) Task 2

Jill used a heart shape (♥) to show a group of 100. She used a sunshine shape (☀) to show a group of 10. She used a star shape (☆) to show 1 object. How could Jill use the same shapes to show the number 304 in two different ways?

standing of special case (a) of the Standard:

Mrs. Johnson purchased a box of 300 pencils. She put them into bags in sets of 10. How many bags did she fill?

To assess the Representations dimension, consider using a familiar visual model for place-value concepts, such as base-ten blocks (see fig. 2a), or a more abstract representation (see fig. 2b).

A grade 3 example

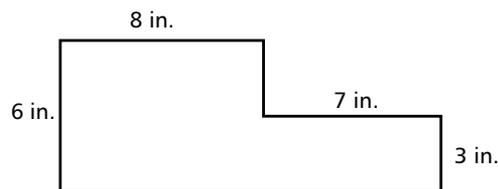
Another major domain of focus in K–grade 5 is Measurement and Data. A specific grade 3 Standard in this domain, clustered under Geometric Measurement, is for students to “relate area to the operations of multiplication and addition” (CCSS 2010, p. 25). Before reading further, try to write an assessment task for this Standard, addressing each of the four dimensions.

When considering tasks related to area, we

FIGURE 3

Use this Representation task to assess students’ understanding of the relationship between area and the operations of addition and multiplication.

Find the area of the following figure. Describe your process.



naturally think of items that fall into the Representations dimension. For example, you might ask students to find the area of a region such as that in figure 3. Results of this task can help you determine if students understand that area is

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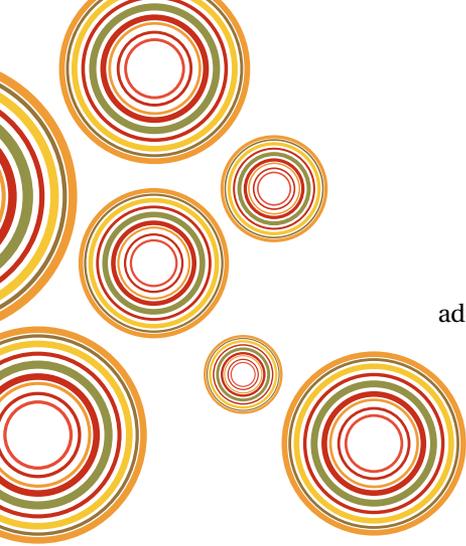
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additive, because a figure can be decomposed into two or more nonoverlapping regions whose individual areas can be found and combined.

To assess the Skills dimension of understanding, you might simply ask students to find the area of a rectangle with a length of 14 meters and a width of 3 meters. Tasks that work “backward” by providing students with the area of a geometric figure, such as a square, and asking them to determine the figure’s dimensions can serve as valuable assessments of students’ understanding of the Properties of area. Given that the area of a square is 49 square feet, students will have to think about the properties of a square, such as the fact that all four sides are equal, and make connections between the multiplication of the side lengths of a square and its area.

Finally, to assess the Uses dimension, a real-life situation such as the following offers an opportunity for students to engage in an open-ended task related to area:

Juan wants to build a rectangular pen in his backyard for his dog, Spot, to run. Juan wants Spot to have 144 square yards of space. If the pen needs to be at least 6 yards wide, what are three possible dimensions of the pen?

A grade 4 example

One of the major domains of focus at the upper elementary school level (grades 3–5), is Number



Simply asking students to explain their reasoning can yield a valuable assessment.

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and Operations–Fractions. At grade 4, a specific standard under this domain (4.NF) is for students to “apply and extend previous understandings of multiplication to multiply a fraction by a whole number” (CCSSI 2010, p. 30). Again, before reading further, try to write an assessment task for this Standard that addresses each of the four dimensions. We immediately thought of the Skills tasks that could be used to assess students’ understanding of fraction multiplication. Common assessment tasks for this mathematical Standard might look like these:

Find the solution to $\frac{1}{2} \times 6$.

Compute $5 \times \frac{3}{4}$.

However, even though many of our students can solve these skill-based problems, they do not necessarily have a well-developed understanding of fraction multiplication. Using SPUR, we can design assessment items to determine if

FIGURE 4

To assess understanding of fraction multiplication in the Properties, Uses, and Representations dimensions, a teacher might use tasks similar to these.

Properties

I have 18 parts, and each part has size $\frac{2}{5}$. What number am I?

Uses

Coach Don asked the girls on his track team to run 15 laps around the track. If each lap is $\frac{1}{4}$ mile, how far did the girls run by the end of practice?

Representations

Let \oplus represent 1 whole. What number is represented by $\square\square\square\square\square$? Justify your response.

students are developing a multidimensional understanding of the concept (see **fig. 4**). As students explain their thinking, as in the Representations task, they give further evidence of the depth of their knowledge.

Aligning assessments with instruction

Assessment plays a crucial role in our work as teachers. NCTM's *Assessment Standards for School Mathematics* (1995) suggests that we ensure that our assessments align with our instruction. We should also use assessment items that reveal useful information about our students' understanding and that can inform our future instruction. SPUR serves as a straightforward guide to analyzing current assessments and creating or modifying future assessments.

Teachers might conduct an investigation of student results on the last classroom assessment by classifying each of the tasks according

to the dimensions of Skills, Properties, Uses, and Representations:

- Was the assessment balanced across the four dimensions? If so, how did your students perform on the tasks in each dimension?
- Do the results suggest that a greater focus of your instruction should be devoted to a particular dimension of understanding?
- If the tasks on your last assessment were not balanced across these four dimensions, how might you modify the tasks to ensure a more balanced assessment of your students' understanding?

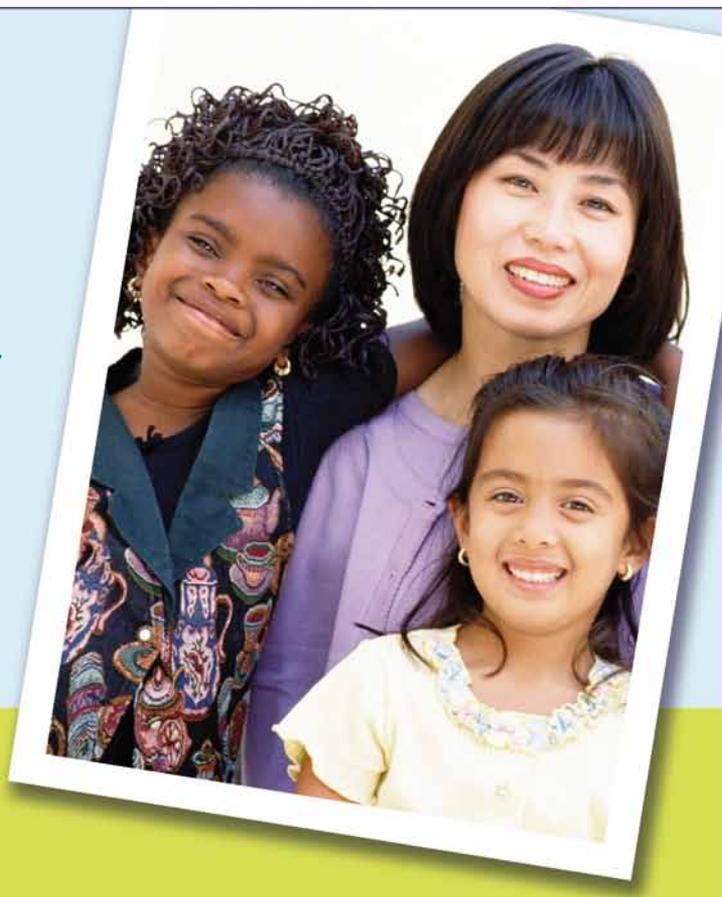


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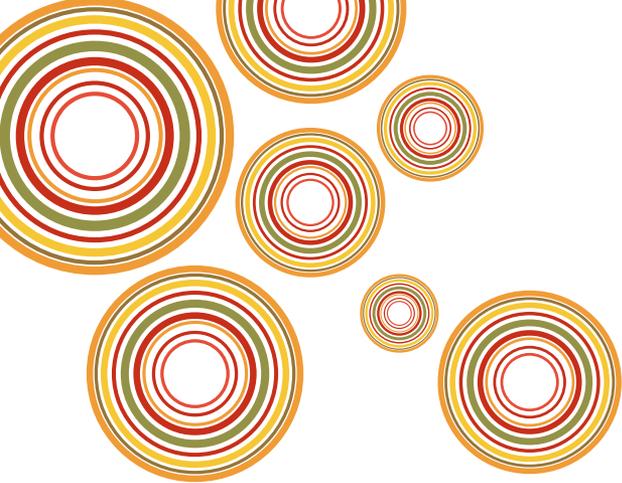
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Our examples of tasks in the four SPUR categories align with particular elementary school standards from the CCSSM document. However, the SPUR framework is applicable at all levels, K–grade 12. As Hirschhorn and his colleagues (1995) indicated, using the SPUR framework with algebra and geometry content at the high school level can guarantee that students will investigate these important concepts from perspectives other than just rote algebraic manipulations or memorized geometric proofs. So, whether you are working from the Common Core State Standards for Mathematics or another set of mathematical learning objectives, it would be beneficial to ask yourself, “What might I expect students to understand about this mathematical topic in each SPUR dimension?” If students can move fluently among their understanding of the topic in the Skills, Properties, Uses, and Representations dimensions, then we would argue that they are developing a robust and connected understanding of mathematics.

REFERENCES

- Bleiler, Sarah K., and Denisse R. Thompson. 2010. “Dimensions of Mathematics Understanding: A Longitudinal Analysis of Primary Students’ Achievement in the United States.” In *The 5th East Asia Regional Conference on Mathematics Education (EARCOME5): In Search of Excellence in Mathematics Education*, edited by Yoshinori Shimizu, Yasuhiro Sekiguchi, and Keiko Hino, pp. 400–7. Tokyo, Japan: Inamoto Printing.
- Centre for Innovation in Mathematics Teaching, International Project on Mathematical Attainment (IPMA) <http://www.cimt.plymouth.ac.uk/projects/ipma/default.htm>.
- Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.
- Hirschhorn, Daniel B., Denisse R. Thompson, Zalman Usiskin, and Sharon L. Senk. 1995.

- “Rethinking the First Two Years of High School Mathematics with the UCSMP.” *Mathematics Teacher* 88 (8): 640–47.
- National Council of Teachers of Mathematics (NCTM). 1995. *Assessment Standards for School Mathematics*. Reston, VA: NCTM.
- . 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- National Mathematics Advisory Panel (NMAP). 2008. *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education.
- National Research Council (NRC). 2001. *Adding It Up: Helping Children Learn Mathematics*, edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell. Washington, DC: National Academies Press.
- Thompson, Denisse R., and Berinderjeet Kaur. 2011. “Using a Multidimensional Approach to Understanding to Assess Students’ Mathematical Knowledge.” In *Assessment in the Mathematics Classroom*, edited by Berinderjeet Kaur and Khoon Yoong Wong, pp. 17–32. Singapore: World Scientific Publishing.
- Usiskin, Zalman. 2007. “The Case of the University of Chicago School Mathematics Project—Secondary Component.” In *Perspectives on the Design and Development of School Mathematics Curricula*, edited by Christian R. Hirsch, pp. 173–82. Reston, VA: National Council of Teachers of Mathematics.



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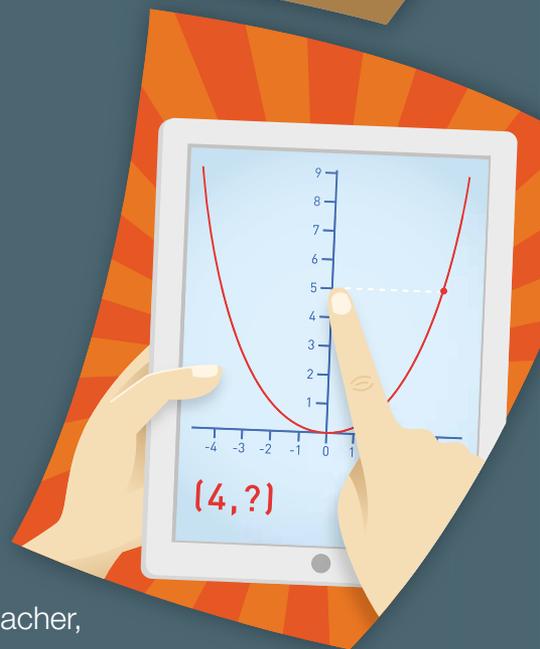


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